Berry Phase and Dissipation of Topological Singularities

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Abstract

We outline a path integral derivation of both the transverse force, the Berry phase, and friction for a vortex from the microscopic fermionic superfluid theory. The derivation manifests transparently the mutual independence of the Berry phase and dissipative terms in the effective vortex action. [1]

I. INTRODUCTION

Vortices in superconductors and He3 superfluids are topological excitations. They determine the global properties such as the stability of the supercurrent carrying states, and have been under intensive theoretical and experimental studies since earlier sixties [1]. The detailed microscopic Bardeen-Cooper-Schrieffer type (BCS) theory for fermionic superfluids is successful and well defined. The derivation of vortex dynamics is, however, non-trivial and less certain. Here we present a path integral derivation of vortex dynamics following the line of geometric methods [2,3]. The advantage of the present method is that in the effective vortex Lagrangian the separation of geometric and dissipative contributions is natural. The important results here are that the transverse force agrees with the one obtained by the Berry phase method [2] and by the total force-force correlation function method [3], and is insensitive to details. The friction is determined by the spectral function of the Hamil-

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tonian, sensitive to details. The large transverse force has been found recently by a direct measurement [4], in consistent with the prediction in Ref. [2,3].

II. DESCRIPTION OF THE BERRY PHASE AND DISSIPATION

In this section we review the description of the Berry phase and dissipation for a particle within a model Hamiltonian, and show how those two quantities appear independently in the effective particle action.

To study the quantum dissipative dynamics of a particle, one may specify the whole Hamiltonian, particle plus environment. In practice, what normally needed is the particle dynamics with the influence from the environment. The environment itself is left unobserved. Hence an accurate and efficient method to integrate out environmental degrees of freedoms is needed. This can be most readily done using the path integral method. In the present paper the approach and notation of Ref. [5] to this question will be followed. In that formulation, the dissipation is produced by an environment consisting of a set of harmonic oscillators. The corresponding total model Hamiltonian may have the following form:

$$H = \frac{1}{2m_v}\left[\mathbf{P} + \mathbf{A}(\mathbf{x})\right]^2 + U(\mathbf{x}) + \sum_j \left[\frac{1}{2m_j}\mathbf{p}_j^2 + \frac{1}{2}m_j\omega_j^2\left(\mathbf{q}_j - \frac{c_j}{m_j\omega_j^2}\mathbf{x}\right)^2\right] . \tag{1}$$

Here $\mathbf{x} = (x, y), \mathbf{P}, \{\mathbf{q}_j\}$, and $\{\mathbf{p}_j\}$ are all two dimensional vectors. The vector potential $\mathbf{A}$ determined by $\nabla \times \mathbf{A} = \mathbf{B}$ corresponds the transverse force $-\dot{\mathbf{x}} \times \mathbf{B}$, with $\mathbf{B}$ in the z-direction. The effect of the dissipative environment is specified by the spectral function

$$J(\omega) \equiv \frac{\pi}{2}\sum_j \frac{c_j^2}{m_j\omega_j}\delta(\omega - \omega_j) . \tag{2}$$

In the present paper, we shall assume the spectral function to have the following form

$$J(\omega) = \eta\omega^s \exp\left\{-\frac{\omega}{\omega_c}\right\} , \tag{3}$$

with ω_c the cutoff frequency whenever needed. In accordance with Ref. [5], $s > 1$ is the superohmic case, $s = 1$ the ohmic case, and $0 \leq s < 1$ the subohmic case. In the ohmic damping case, η is the friction coefficient in the usual Langevin equation.

In the imaginary time formulation the particle dynamics is described by the Euclidean action

$$S = \int_0^{\hbar\beta} d\tau \Bigg[\frac{1}{2} m_v \dot{\mathbf{x}}^2 + i\dot{\mathbf{x}} \cdot \mathbf{A} + U(\mathbf{x}) + \sum_j \left(\frac{1}{2} m_j \dot{\mathbf{q}}_j^2 + \frac{1}{2} m_j \omega_j^2 \left(\mathbf{q}_j - \frac{c_j}{m_j \omega_j^2} \mathbf{x} \right)^2 \right) \Bigg] \ , \tag{4}$$

where $\beta = 1/k_B T$ is the inverse temperature.

Since we are interested in the particle dynamics, keeping other degrees of freedoms of the environment is unnecessary. The integrations over $\{q_j\}$ will then be taken. The resulting effective action for the particle is [5,6]

$$S_{eff}[\mathbf{x}(\tau)] = \int_0^{\hbar\beta} d\tau \left[\frac{1}{2} m_v \dot{\mathbf{x}}^2 + i\dot{\mathbf{x}} \cdot \mathbf{A} + U(\mathbf{x}) \right] + \frac{1}{2} \int_0^{\hbar\beta} d\tau \int_0^{\tau} d\tau' F_{\|}(\tau - \tau') [\mathbf{x}(\tau) - \mathbf{x}(\tau')]^2 \ , \tag{5}$$

with the damping kernel $F_{\|}$ as

$$F_{\|}(\tau) = \frac{1}{\pi} \int_0^{\infty} d\omega \ J(\omega) \frac{\cosh\left[\omega \left(\frac{\hbar\beta}{2} - |\tau| \right) \right]}{\sinh\left[\frac{\omega\hbar\beta}{2} \right]} \ . \tag{6}$$

It is evident that the transverse force corresponds the term $i\dot{\mathbf{x}} \cdot \mathbf{A}$ in the (effective) particle action. It has not been influenced by the dissipative environment, due to the rotational and translational invariances of the coupling between the particle and the environment. This term gives the Berry phase, or the Aharonov-Bohm phase for a charged particle, if moving along a closed trajectory. The influence of the environment is contained in the non-local term. It should be pointed out that in the dissipative dynamics described by Eqs.(5,6) all one needs to know is the spectral function $J(\omega)$. The details of the original environmental Hamiltonian have been suppressed. This implies that the dissipation can be produced by a fermionic environment, an observation important in our following derivation.

III. EFFECTIVE VORTEX ACTION

To find the effective vortex action, we begin with the standard BCS Lagrangian for s-wave pairing in the imaginary time representation. Here the unwanted fermionic degrees of

freedoms will be integrated out, instead of those bosonic ones in section 2. We will only consider neutral superfluids here, but the coupling to electromagnetic fields does no affect our main results. More detailed analysis will be published elsewhere. [7] The model BCS Lagrangian is

$$L_{BCS} = \sum_{\sigma} \psi^{\dagger}_{\sigma}(x,\tau) \left(\hbar\partial_{\tau} - \mu_F - \frac{\hbar^2}{2m}\nabla^2 + V(x) \right) \psi_{\sigma}(x,\tau) \\ -g\psi^{\dagger}_{\uparrow}(x,\tau)\psi^{\dagger}_{\downarrow}(x,\tau)\psi_{\downarrow}(x,\tau)\psi_{\uparrow}(x,\tau) \ , \tag{7}$$

where ψ_σ describes electrons with spin $\sigma = (\uparrow, \downarrow)$, μ_F the chemical potential determined by the electron number density, $V(x)$ the impurity potential, and $x = (x, y, z)$. A vortex at x_v has been implicitly assumed. The partition function is

$$Z = \int \mathcal{D}\{x_v, \psi^{\dagger}, \psi\} \times \exp\left\{ -\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \int d^3x L_{BCS} \right\} \ , \tag{8}$$

with $\beta = 1/k_B T$, and $d^3x = dxdydz$. Inserting the identity in the functional space,

$$1 = \int \mathcal{D}\{\Delta^*, \Delta\} \exp\left\{ -\frac{g}{\hbar}\int_0^{\hbar\beta} d\tau \int d^3x \times \left| \psi_{\downarrow}\psi_{\uparrow} + \frac{1}{g}\Delta(x,\tau) \right|^2 \right\} \ ,$$

into Eq.(8) we have

$$Z = \int \mathcal{D}\{x_v, \psi^{\dagger}, \psi, \Delta^*, \Delta\} \times \exp\left\{ -\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \int d^3x \times \right. \\ \left. \left(\psi^{\dagger}_{\uparrow}, \psi_{\downarrow}\right) (\hbar\partial\tau + \mathcal{H}) \begin{pmatrix} \psi_{\uparrow} \\ \psi^{\dagger}_{\downarrow} \end{pmatrix} - \frac{1}{g}\int_0^{\hbar\beta} d\tau \int d^3x |\Delta|^2 \right\} \ .$$

Here the Hamiltonian is defined as

$$\mathcal{H}(\Delta, \Delta^*) = \begin{pmatrix} H & \Delta \\ \Delta^* & -H^* \end{pmatrix} \ , \tag{9}$$

with $H = -(\hbar^2/2m)\nabla^2 - \mu_F + V(x)$. Integrating out the electron fields $\psi^{\dagger}_{\sigma}$ and ψ_σ first, then the auxiliary(pair) fields Δ under the meanfield approximation, one obtains the partition function for the vortex

$$Z = \int \mathcal{D}\{x_v\} \exp\left\{ -\frac{S_{eff}}{\hbar} \right\} \ , \tag{10}$$

with the effective vortex action

$$\frac{S_{eff}}{\hbar} = -Tr \ln G^{-1} - \frac{1}{\hbar g} \int_0^{\hbar\beta} d\tau \int d^3x |\Delta|^2 \ , \tag{11}$$

where Tr includes internal and space-time indices, and the Nambu-Gor'kov (NG) Green's function G defined by

$$(\hbar\partial_\tau + \mathcal{H}) G(x,\tau;x',\tau') = \delta(\tau-\tau')\delta^3(x-x'), \tag{12}$$

together with the BCS gap equation, or the self-consistent equation,

$$\Delta(x,\tau) = -g \ < \psi_\downarrow(x,\tau)\psi_\uparrow(x,\tau) > \ . \tag{13}$$

A special attention should be paid to the equal time limit of the NG Green's function. [8]

We assume that the vortex is confined to move in a small regime around a point at x_0, which allows a small parameter expansion in terms of the difference between the vortex position x_v and x_0. We look for the long time behavior of vortex dynamics under this small parameter expansion. For the meanfield value of the order parameter, this expansion is

$$\Delta(x,\tau,x_v) = \left(1 + \delta x_v(\tau)\cdot\nabla_0 + \frac{1}{2}(\delta x_v(\tau)\cdot\nabla_0)^2\right)\Delta_0(x,x_0) \ . \tag{14}$$

Here $\delta x_v = x_v - x_0$. In Eq.(14) we have used the fact that when $x_v = x_0$ the vortex is static. The effective vortex action to the same order is, after dropping a constant term,

$$\frac{S_{eff}}{\hbar} = -\frac{1}{2} Tr(G_0\Sigma')^2 + \frac{1}{\hbar g}\int_0^{\hbar\beta} d\tau \int d^3x \times \delta x_v\cdot\nabla_0\Delta_0^* \ \delta x_v\cdot\nabla_0\Delta_0 \ , \tag{15}$$

with

$$\Sigma' = \begin{pmatrix} 0 & \delta x_v\cdot\nabla_0\Delta_0 \\ \delta x_v\cdot\nabla_0\Delta_0^* & 0 \end{pmatrix} \ . \tag{16}$$

Here G_0 is the NG Green's function with $\Delta(\Delta^*)$ replaced by $\Delta_0(\Delta_0^*)$, and the gradient ∇_0 is with respect to x_0.

Before constructing the NG Green's function G_0 we consider the eigenfunctions of the Hamiltonian $\mathcal{H}_0 = \mathcal{H}(\Delta_0,\Delta_0^*)$. The stationary Schrödinger equation, the Bogoliubov-de Gennes (BdG) equation, is

$$\mathcal{H}_0\Psi_k(x) = E_k\Psi_k(x) \ , \tag{17}$$

with

$$\Psi_k(x) = \begin{pmatrix} u_k(x) \\ v_k(x) \end{pmatrix} \ .$$

Since $\mathcal{H}_0$ is hermitian, all its eigenstates form a complete and orthonormal set. Eq.(17) have two useful properties:

$$\mathcal{H}_0\Psi(x) = E\ \Psi(x) \Rightarrow \mathcal{H}_0\overline{\Psi}(x) = -E\ \overline{\Psi}(x) \ , \tag{18}$$

with

$$\overline{\Psi}(x) = \begin{pmatrix} v^*(x) \\ -u^*(x) \end{pmatrix} \ ,$$

and, since $\delta x_v \cdot \nabla_0\mathcal{H}_0 = \Sigma'$, for $k \neq k'$,

$$\int d^3x\Psi_k^\dagger(x)\Sigma'\Psi_{k'}(x) = (E_{k'} - E_k)\delta x_v \cdot \int d^3x\Psi_k^\dagger(x)\nabla_0\Psi_{k'}(x) \ . \tag{19}$$

This equation implies that the two ways of specifying the vortex coordinate, through the trapping potential or through the order parameter, are equivalent.

The NG Green's G_0 can be expressed as

$$G_0(x,\tau;x',\tau') = \sum_{n,k} \frac{-1}{\hbar\beta} \frac{e^{-i\omega_n(\tau-\tau')}}{i\hbar\omega_n - E_k}\Psi_k(x)\Psi_k^\dagger(x'). \tag{20}$$

Here $\omega_n = n\pi/\hbar\beta$, with n odd integers. Assuming the rotational symmetry after the impurity average, a straightforward calculation leads to the following effective vortex action

$$\begin{aligned} S_{eff} = \frac{1}{2}\int_0^{\hbar\beta} d\tau \Bigg[& K\ |\delta x_v(\tau)|^2 + \int_0^\tau d\tau' F_\parallel(\tau-\tau')|\delta x_v(\tau) - \delta x_v(\tau')|^2 \\ & + \int_0^{\hbar\beta} d\tau' F_\perp(\tau-\tau')(\delta x_v(\tau) \times \delta x_v(\tau'))\cdot\hat{z}\Bigg] \ , \end{aligned} \tag{21}$$

with the spring constant in the effective potential,

$$K = \frac{1}{g}\int d^3x|\nabla_0\Delta_0^*(x,x_0)|^2 - \int_0^\infty d\omega\frac{J(\omega)}{\omega} \ , \tag{22}$$

the damping kernel,

$$F_{\parallel}(\tau) = \frac{1}{\pi}\int_0^\infty d\omega J(\omega)\frac{\cosh\left[\omega\left(\frac{\hbar\beta}{2}-|\tau|\right)\right]}{\sinh\left[\omega\frac{\hbar\beta}{2}\right]}\ , \tag{23}$$

and the transverse kernel, in the long time limit, in terms of the virtual transitions,

$$\begin{aligned} F_{\perp}(\tau) = -\partial_{\tau-\tau'}\delta(\tau-\tau')\sum_{k,k'}\int d^3x\int d^3x'\ \hbar(f_k - f_{k'}) \\ \frac{1}{2}\hat{z}\cdot\left(\Psi_k^\dagger(x')\nabla_0\Psi_{k'}(x')\times\nabla_0\Psi_{k'}^\dagger(x)\Psi_k(x)\right)\ , \end{aligned} \tag{24}$$

or in terms of the contribution from each state,

$$\begin{aligned} F_{\perp}(\tau) = -\partial_{\tau-\tau'}\delta(\tau-\tau')\sum_{k}\int d^3x\ \hbar\hat{z}\cdot(f_k\nabla_0 u_k^*(x)\times\nabla_0 u_k(x) \\ -(1-f_k)\nabla_0 v_k^*(x)\times\nabla_0 v_k(x))\ . \end{aligned} \tag{25}$$

To reach Eqs.(21-25), following two identities have also been used:

$$\sum_n \frac{e^{-i\omega_n\delta}}{i\hbar\omega_n - E_k} = \begin{cases} \beta\ f_k\ , & \delta = 0^- \\ -\beta\ (1-f_k)\ , & \delta = 0^+ \end{cases}\ , \tag{26}$$

and

$$\sum_{n-n'}\frac{-1}{\beta}\frac{e^{-i(\omega_n-\omega_{n'})\tau}}{i\hbar(\omega_n-\omega_{n'})-E} = \frac{1}{2}\frac{\cosh\left[\frac{E}{\hbar}\left(\frac{\hbar\beta}{2}-|\tau|\right)\right]}{\sinh\left[E\frac{\beta}{2}\right]}\ , \tag{27}$$

with the Fermi distribution function $f_k = 1/(1+e^{\beta E_k})$, and the spectral function

$$\begin{aligned} J(\omega) = \frac{\pi}{2}\sum_{k,k'}\delta(\hbar\omega - |E_k - E_{k'}|)|f_k - f_{k'}|\times \\ \left|\int d^3x\Psi_k^\dagger(x)\nabla_0\mathcal{H}_0\Psi_{k'}(x)\right|^2\ . \end{aligned} \tag{28}$$

In the following we discuss the implications of K, $F_{\parallel}$, and $F_{\perp}$ one by one, and show that Eq.(21) contains both the dissipative effect and the transverse force, or the Berry phase.

IV. DISCUSSIONS

For the purpose of getting the friction and the transverse force, the precise value of K in Eq.(22)is irrelevant.

The longitudinal kernel $F_{\parallel}$ contains all information on the vortex friction, revealed by the fact that Eqs.(21,23) are identical to the nonlocal action of Eqs.(5,6). The friction is determined by the low frequency behavior of the spectral function $J(\omega)$, Eq.(28).

If we classify the eigenstates of the BdG equation, Eq.(17), according to core(localized) and extended states, the dissipation comes from all three parts: core-to-core, core-to-extended (or extended-to-core), and extended-to-extended transition rates. Those rates are added up, because they are all positive, which differs from the situation for the transverse kernel.

For a clean superfluid with no impurities, it is straightforward to find the damping corresponds to super-Ohmic cases. The dissipation is weak in this case. The main effect is to renormalize the vortex mass. [7]

The presence of impurities mixes up all the clean limit eigenstates, and generates a quasi-continuous core energy spectrum after the impurity average. A perturbative calculation shows that it is indeed the case, and the spectral function becomes Ohmic. [7] This suggests that in the clean limit the dissipation is super-Ohmic, and turns into Ohmic in the dirty limit.

In the long time limit there are two equivalent forms for the transverse kernel $F_{\perp}$: The virtual transition expression, Eq.(24), and the individual state contribution, Eq.(25). Their equivalence has already been discussed in Ref. [9] with the aid of conservation laws. Eq.(24) can be again casted into various equivalent forms because of the cancelations among those virtual transitions. For example, at zero temperature only two core states closest to the Fermi surface, one below '–' and one above '+', will be involved:

$$F_{\perp} = -\partial_{\tau-\tau'}\delta(\tau-\tau')\int d^3x \int d^3x'\, \hbar(f_- - f_+) \\ \hat{z}\cdot\left(\Psi_-^{\dagger}(x')\nabla_0\Psi_+(x')\times\nabla_0\Psi_+^{\dagger}(x)\Psi_-(x)\right) . \tag{29}$$

In this case it is a statement of the spectral flow [9]. One may also cast it into the forms of core-to-extended, or extended-to-extended transitions.

In Eq.(25), the counting of individual state contributions is expressed as an area integral

of the momentum commutator. It can be expressed as the line integral of the momentum density far away from the core,

$$F_\perp = i\partial_{\tau-\tau'}\delta(\tau-\tau')\oint dx\cdot j(x-x_0)\ , \tag{30}$$

with the momentum density $j(x)\equiv i\hbar\lim_{x\to x'}\frac{1}{2}(\nabla-\nabla')\rho_1(x,x')$, and the one-body density matrix

$$\rho_1(x,x') = \sum_k \left((f_k u_k^*(x)u_k(x') - (1-f_k)v_k^*(x)v_k(x')\right)\ . \tag{31}$$

Then $F_\perp$ corresponds the term, the Berry phase $ih(\rho_s/2)\int d\tau\ \delta\dot{x}_v\cdot(\delta x_v\times\hat{z})/2$, in the effective vortex action Eq.(21). Here ρ_s is the superfluid number area density. The immediate conclusion, as drawn in Ref. [3], is that localized core states do not contribute, because both the total momentum density at the core x_0 and the momentum density of core states far away from x_0 are zero. Thus we have two complete different ways to evaluate the transverse force: via virtual transitions at the core, or a counting of momentum density for each state far away from the core. This is similar to the evaluation of the quantum Hall conductance: the two methods, through either edge or bulk states, are equivalent.

The insensitive to details such as impurities is most transparent from Eq.(25) or (31): In the one-body density matrix both the electron number density and the phase $\theta(x-x_v)$ are all insensitive to details, if the localization effect caused by impurities is negligible. Here the phase θ is defined through the order parameter $\Delta(x,t,x_v)\to|\Delta|e^{iq\theta(x-x_v)}$, with $q=\pm1$ describing the vorticity along the $\hat{z}$ direction and $\theta(x)=\arctan(y/x)$.

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